IJISET - International Journal of Innovative Science, Engineering & Technology, Vol. 3 Issue 8, August 2016
ISSN (Online) 2348 – 7968 | Impact Factor (2015) - 4.332
www.ijiset.com

Laplace and Sumudu Transforms and Their Application

Janki Vashi[1] and M. G. Timol[2]

1. A S H Department, Bhagwan Mahavir Polytechnic, Surat.
2. Department of mathematics, Veer Narmad South Gujarat University, Surat.

Abstract:

Laplace transform is a very powerful mathematical tool applied in various areas of engineering and science. Although not so well-known as Laplace transform, the Sumudu transform, proposed in the early 1990s, has some interesting advantages over other integral transforms especially the 'unity' feature which could provide convenience when solving differential equations. I this paper will discuss the applications of Laplace transform and Sumudu transform in the area of physics followed by the application to electric circuit. Also the results by these two transforms are compared.

Key words:

Laplace transform, Sumudu transform

1. Introduction:

Laplace transform

Laplace transform is an integral transform method is particularly useful in solving linear ordinary differential equation. In order for any function of time f(t) to be Laplace transformable, it must satisfy the following Dirichlet conditions[1]:

- f(t) must be piecewise continuous which means that it must be single valued but can have a finite number of finite isolated discontinuies for t>0.
- f(t) must be exponential order which means that f(t) must remain less than $se^{-a_0 t}$ as t approaches ∞ where S is a positive constant and a_0 is a real positive number.

If there is any function which satisfies the Dirichlet conditions, then,

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st} f(t)dt, \quad for\ s > 0$$

Some important properties of Laplace transform:

Property	Signal	Transform
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$
Shifting in the S-domain	$e^{t_0 s}x(t)$	$X(s - s_0)$
Time scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$x^*(s^*)$
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$
Differentiation in t-domain	$\frac{d}{dt}x(t)$	$sX(s)$
Differentiation in s-domain	$-tx(t)$	$\frac{d}{ds}X(s)$
Integration in t-domain	$\int_{-t}^{\infty} x(\tau)d(\tau)$	$\frac{1}{s}X(s)$

Table 1.1: Properties of Laplace transform

Sumudu transform:

A new integral transform called the Sumudu transform is introduced in 1993 by G.K.watugala. this transform has many interesting properties. This transform is defined over the set of functions

A = {f (t): ∃ M , $\tau_1, \tau_2 > 0, |f(t)| < Me^{\frac{t}{\tau}}$, if t ∈ $(-1)^j \times [0, \infty)$}

The Sumudu transform of time function F(t) denoted by S [F(t) : u] or simply F(u) is defined as follow:

F(u) = $\frac{1}{u}\int_0^\infty e^{-\frac{t}{u}}.f(t)dt$ (0<t<∞)

Where u is a parameter and it may be real or complex and it is independent of t.

The inversion formula for Sumudu transform is given by

$$S^{-1}[G(s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}G\left(\frac{1}{s}\right)\frac{ds}{s}$$

Operational properties of Sumudu transform:

Formula	**Remarks**
$f(Ct) = G(Cu)$	Scaling by C
$\lim_{t\to 0} f(t) = \lim_{u\to 0} G(u)$	Limit at origin
$S[f(at)] = G(au)$	First scale preserving theorem
$S\left[t\frac{d}{dt}f(t)\right] = u.\frac{d}{du}G(u)$	Second scale preserving theorem
$S[e^{at}f(t)] = \frac{1}{1-au}G(\frac{u}{1-au})$	First shifting theorem
$S[f(t-a)H(t-a)] = e^{-au}G(u)$	Second shifting theorem
$S\left[\frac{1}{t}\int_0^t f(t)dt\right] = \frac{1}{u}\int_0^t f(v)dv$	Average preserving
$\lim_{u\to\pm\infty} G(u) = \lim_{t\to\pm\infty} f(t)$	Final value theorem
$S(f(t)) = \frac{\int_0^{\frac{T}{u}} f(ut)e^{-t}dt}{1-e^{-\frac{T}{u}}}$	Sumudu transform of a T-periodic function
$S(u) = \frac{F(\frac{1}{u})}{u}$ and $F(s) = \frac{S(\frac{1}{s})}{s}$	Duality with Laplace transform
$S(f*g) = u.S(f(t)).S(g(t))$ $(f*g) = \int_0^t f(\tau)g(t-\tau)d\tau$	Sumudu convolution theorem

Table 1.2: Properties of Sumudu transform

2. Comparison between Laplace and Sumudu transform:

The Sumudu transform is a simple variant of the Laplace transfom. The Sumudu transform is essentially identical with the Laplace. Given an initial *f*(*t*), its Laplace transform $F(s)$ can be translated into the Sumudu transform $F_s(u)$ of *f* by means of the relation

$$S(u) = \frac{F\left(\frac{1}{u}\right)}{u}$$

And its inverse,

ISSN (Online) 2348 – 7968 | Impact Factor (2015) - 4.332
www.ijiset.com

$$F(s) = \frac{S\left(\frac{1}{s}\right)}{s}$$

Every property proved of the Laplace transform may routinely be turned into a corresponding property of the Sumudu transform (and again vice versa). This proves the essential identity of the two transforms.

3. Application

Application in physics

Let us consider a beam of length of l and uniform cross section parallel to the YZ plane so that the normal deflection w(x,t) is measured downward if the axis of the beam is towards X-axis. The basic equation defining this phenomenon is as given below:

$$EI\frac{d^4\omega}{dx^4} - m\omega^2 w = 0 \qquad (1)$$

Where E is Young's modulus of elasticity; I is the moment of inertia of the cross section with respect to the y-axis; m is the mass per unit length; and w is the angular frequency.

Now, rewriting equation (1) by setting $\alpha^4 = \frac{m\omega^2 w}{EI}$, we obtain

$$\frac{d^4\omega}{dx^4} - \alpha^4\omega = 0 \qquad (2)$$

Solutions by Laplace transform:

Now, applying Laplace transform to equation (2)

$$s^4 f(s) - s^3F(+0) - s^2F'(+0) - sF''(+0) - sF'''(+0) - \alpha^4 f(s) = 0$$

The boundary conditions for this problem are:

$$F(+0) = 0; F(+l) = 0; F''(+0) = 0; F''(+l) = 0$$

Hence, we obtain,

$$f(s) = s^2F'(+0) + \frac{F'''(+0)}{s^4 - \alpha^4}$$

The inverse Laplace transform gives,

$$\omega = \frac{F'(+0)}{2\alpha} sinh\alpha x + sin\alpha x + \frac{F'''(+0)}{2\alpha} sinh\alpha x - sin\alpha x$$

That is

$$\omega = A_1 sinh\alpha x + A_2 sin\alpha x \qquad (3)$$

When $x = l; A_1 sinh\alpha l + A_2 sin\alpha l = 0; A_1 sinh\alpha l - A_2 sin\alpha l = 0$

These are satisfied if $A_1 = A_2 = 0$ $i.e. sinh\alpha l = sin\alpha l = 0$. This will give, $\alpha l = n\pi$, for integral values of n. Hence, $A_1 = 0$ and A_2 is undetermined and the resulting vibrations are:

$W_n = A_n sin\left(\frac{n\pi x}{l}\right)$ and $\omega_n = \frac{\pi^2 n^2}{l^2}\sqrt{\frac{EI}{m}}$

Here, if n=1, it represent the fundamental vibration and if n=2 the first harmonic and so on.

Solution by Sumudu transform:

Now , we apply Sumudu transform and boundary conditions on equation (2) we get,

$$\frac{F(u)}{u^4} - \frac{F'(0)}{u^3} - \frac{F'''(0)}{u} - \alpha^4 F(u) = 0$$

Simplifying we get,

$$F(u) = \frac{uF'(0) + u^3F''(0)}{1 - \alpha^4 u^4}$$

The inverse Sumudu transform gives,

$$\omega = \frac{F'(+0)}{2\alpha} sinh\alpha x + sin\alpha x + \frac{F'''(+0)}{2\alpha} sinh\alpha x - sin\alpha x$$

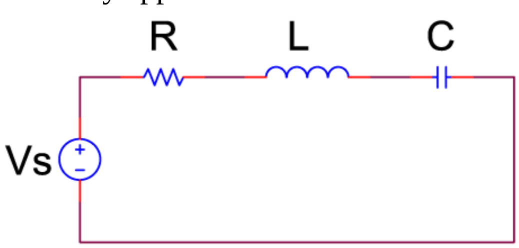
That is

$$\omega = A_1 sinh\alpha x + A_2 sin\alpha x$$

Above solution is same as equation (3), which we have obtained by applying Laplace transform.

Application in Electric Circuit Theory

Let us consider a series RLC circuit as shown in Fig.1. to which a d.c voltage V_s is suddenly applied.



Now, applying Kirchhoff's Law to the circuit we have,

$$Ri + L\frac{di}{dt} + \frac{1}{c}\int idt = V_s \quad (4)$$

Differentiating both sides,

$$\frac{d^2 i}{dt^2} + \left(\frac{R}{L}\right)\frac{di}{dt} + \left(\frac{1}{LC}\right) i = 0 \quad (5)$$

Solution by Laplace transform:
Now, applying Laplace transform to this equation, let as assume that the solution of this equation is

$i(t) = Ke^{st}$ where K and s are constant which may be real, imaginary or complex.

Now, from equation (5),

$LKs^2e^{st} + RKe^{st} + \frac{1}{c}Ke^{st} = 0$ which on simplification gives,

$$s^2 + \left(\frac{R}{L}\right)s + \frac{1}{LC} = 0$$

The roots of this equation would be

$$s_1, s_2 = \frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad (6)$$

The general solution of the differential equation is thus,

$i(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$ where K_1 and K_2 are determined from the initial conditions.

Now, if we define , α = Damping Coefficient $= \frac{R}{2L}$ and natural frequency, $\omega_n = \sqrt{\frac{1}{LC}}$ which is also known as undamped natural frequency or resonant frequency.

Thus roots are : $s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_n^2}$

The final form of solution depends on whether

$$\frac{R^2}{4L^2} > \frac{1}{LC};\ \frac{R^2}{4L^2} = \frac{1}{LC}\ and\ \frac{R^2}{4L^2} < \frac{1}{LC}$$

Solution by Sumudu transform:
Now, we apply Sumudu transform on equation (5)

$$LK\frac{1}{u^2}e^{st} + RK\frac{1}{u} + \frac{1}{c}Ke^{st} = 0$$

Simplification gives,

$$u^2 + RCu + LC = 0$$

The roots of this equation would be

$$s_1, s_2 = -\frac{RC}{2} \pm \sqrt{\frac{R^2L^2}{4} - LC}$$

The general solution of the differential

equation is thus,

$i(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t}$ where K_1 and K_2 are determined from the initial conditions.

4. Conclusion

The Sumudu transform and Laplace transform along with the some of their important operational properties are introduced. Sumudu and Laplace transforms are used to solve application in physics and Circuit theory. The result confirms that the Sumudu transform technique is simple and powerful tool. It is anticipated that Sumudu transform method can be used to find analytical solution of much complex differential equation of higher order and system of differential equations.

5. References

[1] Siddiqui A.M, Rana M.A. and Ghori Q.K., Application of Sumudu Transform to Newtonian Fluid Problems, Adv. Studies Theor. Phy., Vol. 4,no.1,(2010),21-29

[2] Belgacem F.M. and Karaballi A., Sumudu transform Fundamental Properties investigation and Applications,journel of Applied Mathematics and Stochastis Analysis,(2006),1-23

[3] Watugala G.K, Sumudu transform – a new integral transform to solve differential equations and control engineering problems, Math.Engg. in Industry,6(4)(1998),319-329.

[4] Belgacem F.M and Karaballi A. and Kalla S.L., analytical investigation of the Sumudu transform and applications to integral production equations,Mathematical problems in engineering ,3 (2003),103-118

[5] Asiru M.A., Further properties of Sumudu transform and its applications, international journal of mathematical Ed. in Sci. and technology, 6(2010)

[6] Watugala G.K, Sumudu transform – an integral transform to solve differential equations and control engineering problems, international journal of mathematical Ed. in Sci. and technology,24(1993),35-42

[7] Li Kejie, Xie Yanzho, A Brief introduction of Sumudu transform and comparison with other Integral transform,IEEE,(2011)